Unique continuation of Schrödinger-type equations for $\bar{\partial}$ II

Yifei Pan and Yuan Zhang

Abstract

In this paper, we extend our earlier unique continuation results [8] for the Schrödinger-type inequality $|\bar{\partial}u| \le V|u|$ on a domain in $\mathbb{C}^n$ by removing the smoothness assumption on solutions $u = (u_1, \ldots, u_N)$. More specifically, we establish the unique continuation property for $W^{1,1}_{loc}$ solutions when the potential $V \in L^p_{loc}$, $p > 2n$; and for $W^{1,2n+\epsilon}_{loc}$ solutions when $V \in L^{2n}_{loc}$ with $N = 1$ or $n = 2$. Although the unique continuation property fails in general if $V \in L^p_{loc}, p < 2n$, we show that the property still holds for $W^{1,1}_{loc}$ solutions when V is a small constant multiple of $\frac{1}{|z|}$.

1 Introduction

Let Ω be a domain in $\mathbb{C}^n$. Suppose $u : \Omega \to \mathbb{C}^N$ with $u \in W^{1,1}_{loc}(\Omega)$, and satisfies the following Schrördinger-type inequality

$$|\bar{\partial}u| \le V|u| \quad \text{a.e. on} \quad \Omega, \tag{1.1}$$

for some nonnegative locally Lebesgue integrable function V on Ω. This paper is a continuation of an earlier work [8] of the same authors on the unique continuation property for (1.1). Specifically, we investigate whether any Sobolev function u satisfying (1.1) vanishes identically if u vanishes to infinite order in the L^1 sense at some $z_0 \in \Omega$. Here for $q \ge 1$, $u \in L^q_{loc}(\Omega)$ is said to vanish to infinite order (or, be flat) in the L^q sense at a point $z_0 \in \Omega$, if for all $m \ge 1$,

$$\lim_{r \to 0} r^{-m} \int_{|z-z_0|<r} |u(z)|^q dv_z = 0.$$

While Example 1 indicates the general failure of the unique continuation property for L^p_{loc} potentials with $p < 2n$, it was shown in [8] the property holds for smooth solutions to (1.1) with L^p_{loc} potentials, $p > 2n$, and L^{2n}_{loc} potentials if either $N = 1$ or $n = 2$. Although the smooth category remains of primary interest, our goal in this paper is to extend the unique continuation property to general Sobolev solutions.

We first show the unique continuation property holds for $W^{1,1}_{loc}$ solutions to (1.1) when the potential $V \in L^p_{loc}$ for some $p > 2n$. In particular, this generalizes a unique continuation result of Bell and Lempert in [2] for bounded potentials. See also a very recent result by Shi [9] under some stronger assumptions on the potentials and the regularity of solutions.

Theorem 1.1. *Let Ω be a domain in $\mathbb{C}^n$. Suppose $u : \Omega \to \mathbb{C}^N$ with $u \in W^{1,1}_{loc}(\Omega)$, and satisfies $|\bar{\partial}u| \le V|u|$ a.e. on Ω for $V \in L^p_{loc}(\Omega)$, $p > 2n$. If u vanishes to infinite order in the L^1 sense at some $z_0 \in \Omega$, then u vanishes identically.*

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Similar to the approach in [8], the proof utilizes a complex polar coordinate formula in Lemma 3.1 to convert the unique continuation problem in higher dimensional domains to a known one-dimensional result in [7] (see Theorem 2.1). While the majority of the work towards the proof was already established in [8], unlike the smooth case, the flatness of Sobolev solutions and the inequality (1.1) do not extend immediately to their sliced counterparts. The novelty in the proof is to show that the minimal $W^{1,1}_{loc}$ Sobolev regularity of u is sufficient in ensuring the sliced solutions to satisfy all the assumptions in Theorem 2.1. In fact, the assumptions $u \in W^{1,1}_{loc}$ and $V \in L^p_{loc}, p > 2n$ here actually imply $u \in W^{1,p}_{loc}$, following a standard boot-strap argument. This, combined with Lemma 3.3, transforms a $W^{1,1}_{loc}$ solution of (1.1) to a family of $W^{1,2}_{loc}$ solutions to some Schrödinger-type inequalities along almost every complex radial direction. Moreover, we establish in Lemma 3.4 an equivalence between the L^1 flatness of $W^{1,p}_{loc}$ functions and a much stronger geometric flatness as in (3.6), which further passes the flatness onto the sliced solutions.

When the potential $V \in L^{2n}_{loc}$, we obtain the following two unique continuation results: one for the case $N = 1$ (namely, u is a scalar function), and the other for the case $n = 2$. The integrability assumption of V here is sharp, as indicated by Example 1.

Theorem 1.2. *Let Ω be a domain in $\mathbb{C}^n$. Suppose $u : \Omega \to \mathbb{C}$ with $u \in W^{1,2n+\epsilon}_{loc}(\Omega)$ for some $\epsilon > 0$, and satisfies $|\bar{\partial} u| \leq V|u|$ a.e. on Ω for some $V \in L^{2n}_{loc}(\Omega)$. If u vanishes to infinite order in the L^1 sense at some $z_0 \in \Omega$, then u vanishes identically.*

Theorem 1.3. *Let Ω be a domain in $\mathbb{C}^2$. Suppose $u : \Omega \to \mathbb{C}^N$ with $u \in W^{1,2n+\epsilon}_{loc}(\Omega)$ for some $\epsilon > 0$, and satisfies $|\bar{\partial} u| \leq V|u|$ a.e. on Ω for some $V \in L^4_{loc}(\Omega)$. If u vanishes to infinite order in the L^1 sense at some $z_0 \in \Omega$, then u vanishes identically.*

We have to assume $u \in W^{1,2n+\epsilon}_{loc}$ in both theorems above to apply Lemma 3.3 and Lemma 3.4. This is essentially due to the failure of the boot-strap argument to improve the Sobolev regularity of solutions when $V \in L^{2n}_{loc}$. It would be desirable to know whether this regularity assumption on u can be weakened further.

On the other hand, it is worth noting that for the Laplacian Δ, the unique continuation property for $W^{2,2}_{loc}(\Omega)$ solutions of the inequality

$$|\Delta u| \leq V|\nabla u| \quad \text{on} \quad \Omega \subset \mathbb{R}^d$$

with $V \in L^d_{loc}(\Omega)$ holds when $d = 2, 3, 4$, but fails in general when $d \geq 5$. See the works of Chanillo-Sawyer [4] and Wolff [10, 11]. In contrast, Theorem 1.2 shows that, in the critical $V \in L^{\dim_{\mathbb{R}} \Omega}_{loc}(\Omega)$ case, the unique continuation property for $\bar{\partial}$ is dimension-independent, highlighting a significant difference from the case of Δ.

Finally, we explore a special case where V is a constant multiple of $\frac{1}{|z|}$. Note that $\frac{1}{|z|} \notin L^{2n}_{loc}$ near 0. Although as in the L^{2n}_{loc} potential case the boot-strap argument does not improve the Sobolev regularity of solutions near 0 in general, thanks to Lemma 4.4, the additional flatness of u at 0 allows us to eventually push u to fall in $W^{1,q}_{loc}$ for all $q < \infty$. In view of this, the $N = 1$ case in the following theorem is simply a direct consequence of [8, Theorem 5.1] for $W^{1,2}_{loc}$ solutions. For $N \geq 2$ case, one can use a similar approach as in Theorem 1.1 to weaken the smoothness assumption of u in [8, Theorem 1.5] to merely $W^{1,1}_{loc}$ regularity. It should also be mentioned that when $N \geq 2$, the unique continuation property fails in general if C is large, see Example 2.

Theorem 1.4. *Let Ω be a domain in $\mathbb{C}^n$ and $0 \in \Omega$. Let $u : \Omega \to \mathbb{C}^N$ with $u \in W^{1,1}_{loc}(\Omega)$, and satisfy $|\bar{\partial} u| \leq \frac{C}{|z|}|u|$ a.e. on Ω. Assume u vanishes to infinite order in the L^1 sense at $0 \in \Omega$.*
1). If $N = 1$, then u vanishes identically.
2). If $N \geq 2$ and $C < \frac{1}{4}$, then u vanishes identically.

2 Known results and examples

In this section, we list a few known unique continuation properties for (1.1) that will be used in our paper, along with counter-examples for certain types of potentials. For clarification, $u = (u_1, \dots, u_N) \in W^{1,1}_{loc}(\Omega)$ is said to satisfy the inequality $|\bar\partial u| \le V|u|$ a.e. on Ω if

$$|\bar\partial u| := \left(\sum_{j=1}^{n}\sum_{k=1}^{N} |\bar\partial_j u_k|^2\right)^{\frac12} \le V\left(\sum_{k=1}^{N}|u_k|^2\right)^{\frac12} := V|u| \ \text{ a.e. on } \Omega.$$

It is worth pointing out that, every $u \in W^{1,1}_{loc}(\Omega)$ satisfying $|\bar\partial u| \le V|u|$ a.e. on Ω for some scalar function $V \in L^p_{loc}(\Omega)$ is a weak solution to a Schrödinger-type equation of $\bar\partial$ below

$$\bar\partial u = u\mathcal{V} \ \text{ on } \ \Omega$$

for some $N \times N$ matrix-valued $(0,1)$ form $\mathcal{V} = (\mathcal{V}_{jk}) \in L^p_{loc}(\Omega)$, simply by letting $\mathcal{V}_{jk} := \frac{\overline{u_j}\bar\partial u_k}{|u|^2}$.

Theorem 2.1. *[7] Let Ω be a domain in $\mathbb{C}^n$. Suppose $u : \Omega \to \mathbb{C}^N$ with $u \in W^{1,2}_{loc}(\Omega)$, and satisfies $|\bar\partial u| \le V|u|$ a.e. on Ω for some $V \in L^2_{loc}(\Omega)$.*
1). The weak unique continuation holds: if u vanishes in an open subset of Ω, then u vanishes identically.
2). If $n = 1$, then the (strong) unique continuation holds: if u vanishes to infinite order in the L^2 sense at some $z_0 \in \Omega$, then u vanishes identically.

The unique continuation to (1.1) fails in general if the potential does not belong to L^{2n}_{loc}.

Example 1. *For each $1 \le p < 2n$, let $\epsilon \in (0, \frac{2n}{p} - 1)$ and consider*

$$|\bar\partial u| \le V|u| := \frac{\epsilon}{2|z|^{\epsilon+1}}|u| \quad \text{on} \quad B_1 \subset \mathbb{C}^n.$$

Note that $V \in L^p(B_1)$, and $u_0 = e^{-\frac{1}{|z|^\epsilon}}$ is a nontrivial smooth solution to the above equation that vanishes to infinite order at 0.

Despite Example 1, the unique continuation property can still be expected for some special forms of potentials not in L^{2n}_{loc}, for instance, when the potential is a multiple of $\frac{1}{|z|}$.

Theorem 2.2. *[8, Theorem 5.1] Let Ω be a domain in $\mathbb{C}^n$ and $0 \in \Omega$. Let $u : \Omega \to \mathbb{C}$ with $u \in W^{1,2}_{loc}(\Omega)$, and satisfies $|\bar\partial u| \le \frac{C}{|z|}|u|$ a.e. on Ω for some constant $C > 0$. If u vanishes to infinite order in the L^2 sense at 0, then u vanishes identically.*

Theorem 2.3. *[8, Theorem 6.1] Let Ω be a domain in $\mathbb{C}$ and $0 \in \Omega$. Let $u : \Omega \to \mathbb{C}^N$ with $u \in W^{1,2}_{loc}(\Omega)$, and satisfy $|\bar\partial u| \le \frac{C}{|z|}|u|$ a.e. on Ω for some positive constant $C < \frac14$. If u vanishes to infinite order in the L^2 sense at 0, then u vanishes identically.*

In particular, when $N \ge 2$ and C is large, the above unique continuation property no longer holds in general, as indicated by an example below of the first author and Wolff [6], or [1] by Alinhac and Baouendi.

Example 2. *Let $v_0 : \mathbb{C} \to \mathbb{C}$ be the nontrivial smooth scalar function constructed in [6] that vanishes to infinite order at 0 and satisfies $|\triangle v_0| \le \frac{C^\sharp}{|z|}|\nabla v_0|$ on $\mathbb{C}$ for some constant $C^\sharp > 0$. Letting $u_0 := (\partial \Re v_0, \partial \Im v_0)$, then $u_0 : \mathbb{C} \to \mathbb{C}^2$ is smooth, vanishes to infinite order at 0, and satisfies $|\bar\partial u_0| \le \frac{C^\sharp}{2|z|}|u_0|$ on $\mathbb{C}$.*

In the case when the source dimension $n = 1$, the unique continuation property holds even when the potentials take on the following hybrid forms involving both powers of $\frac{1}{|z|}$ and Lebesgue functions. Note that none of these potentials below belongs to L^2_{loc}.

Theorem 2.4. *[8, Theorem 5.5] Let Ω be a domain in $\mathbb{C}$ containing 0 and $1 < \beta < \infty$. Suppose $u : \Omega \to \mathbb{C}$ with $u \in W^{1,2}_{loc}(\Omega)$, and satisfies*

$$|\bar\partial u| \le |z|^{-\frac{\beta-1}{\beta}} V|u| \quad a.e. \ on \ \ \Omega$$

for some $V \in L^{2\beta}_{loc}(\Omega)$. If u vanishes to infinite order in the L^2 sense at 0, then u vanishes identically.

Theorem 2.5. *[8, Theorem 7.1] Let Ω be a domain in $\mathbb{C}$ and $0 \in \Omega$. Suppose $u : \Omega \to \mathbb{C}^N$ with $u \in W^{1,2}_{loc}(\Omega)$, and satisfies*

$$|\bar\partial u| \le |z|^{-\frac{1}{2}} V|u| \ \ a.e. \ on \ \ \Omega$$

for some $V \in L^4_{loc}(\Omega)$. If u vanishes to infinite order in the L^2 sense at 0, then u vanishes identically.

3 Properties of sliced functions

As will be seen in the next section, we shall slice Sobolev solutions to (1.1) and the potential along almost all complex one-dimensional radial directions. The key idea to justify this approach lies in the following complex polar coordinate formula, whose proof can be found, for instance, in [8, Lemma 4.2]. Denote by S^{2n-1} the unit sphere in $\mathbb{C}^n$. Let B_r be the open ball centered at 0 of radius r in $\mathbb{C}^n$, and D_r be the open disk centered at 0 of radius r in $\mathbb{C}$.

Lemma 3.1. *Let $u \in L^1(B_r)$. Then for a.e. $\zeta \in S^{2n-1}$, $|w|^{2n-2}u(w\zeta)$ as a function of $w \in D_r$ is in $L^1(D_r)$, with*

$$\int_{|z|<r} u(z)dv_z = \frac{1}{2\pi}\int_{|\zeta|=1}\int_{|w|<r}|w|^{2n-2}u(w\zeta)dv_w dS_\zeta.$$

Corollary 3.2. *Let $u \in L^p(B_r)$ for some $p > n$ $(= \dim_{\mathbb{C}} B_r)$. Then given a.e. $\zeta \in S^{2n-1}$, $v(w) := u(w\zeta)$ as a function of $w \in D_r$ is in $L^q(D_r)$ for all $1 \le q < \frac{p}{n}$. In particular, if $u \in L^p(B_r)$ for some $p > 2n$, then for a.e. $\zeta \in S^{2n-1}$, $v \in L^2(D_r)$.*

Proof. by Lemma 3.1

$$\int_{|z|<r} |u(z)|^p dv_z = \frac{1}{2\pi}\int_{|\zeta|=1}\int_{|w|<r}|w|^{2n-2}|u(w\zeta)|^p dv_w dS_\zeta.$$

Then for a.e. $\zeta \in S^{2n-1}$,

$$\int_{|w|<r}|w|^{2n-2}|u(w\zeta)|^p dv_w < \infty. \tag{3.1}$$

Making use of Hölder's inequality

$$\begin{aligned}\int_{|w|<r} |v(w)|^q \, dv_w &= \int_{|w|<r} |w|^{\frac{(2n-2)q}{p}} |v(w)|^q \cdot |w|^{-\frac{(2n-2)q}{p}} dv_w \\ &\le \left(\int_{|w|<r} |w|^{2n-2} |u(w\zeta)|^p dv_w \right)^{\frac{q}{p}} \left(\int_{|w|<r} |w|^{-\frac{(2n-2)q}{p} \frac{p}{p-q}} dv_w \right)^{\frac{p-q}{p}} \\ &= \left(\int_{|w|<r} |w|^{2n-2} |u(w\zeta)|^p dv_w \right)^{\frac{q}{p}} \left(\int_{|w|<r} |w|^{-\frac{(2n-2)q}{p-q}} dv_w \right)^{\frac{p-q}{p}}.\end{aligned}$$

Since $q < \frac{p}{n}$, we have $\frac{(2n-2)q}{p-q} < 2$ and thus $\int_{|w|<r} |w|^{-\frac{(2n-2)q}{p-q}} dv_w < \infty$. This, combined with (3.1), proves the corollary. □

In order to convert the unique continuation property in the higher source dimensional case to the complex one-dimensional case where Theorem 2.1 can be applied, we first establish Lemma 3.3 below. This allows us to obtain sufficient regularity for the sliced functions when $u \in W^{1,p}_{loc}, p > 2n$. As demonstrated in Example 3 in Section 4, the lemma does not hold for general $W^{1,2n}$ functions.

Lemma 3.3. *Suppose $u \in W^{1,p}(B_r)$ for some $p > 2n$ $(= \dim_{\mathbb{R}} B_r)$. Then for a.e. $\zeta \in S^{2n-1}$, $v(w) := u(w\zeta)$ as a function of $w \in D_r$ belongs to $W^{1,2}(D_r)$. Moreover,*

$$\nabla v(w) = \zeta \cdot \nabla u(w\zeta), \quad w \in D_r \tag{3.2}$$

in the sense of distributions.

Proof. We only need to show (3.2). In fact, since $u \in W^{1,p}(B_r)$, $p > 2n$, if (3.2) holds, we can apply Corollary 3.2 to u and ∇u respectively, and obtain $v \in W^{1,2}(D_r)$.

Let $u_j \in C^\infty(B_r) \cap W^{1,p}(B_r)$ be such that $u_j \to u$ in the $W^{1,p}(B_r)$ norm and $v_j(w) := u_j(w\zeta), w \in D_r$. Then

$$\nabla v_j(w) = \zeta \cdot \nabla u_j(w\zeta), \quad w \in D_r. \tag{3.3}$$

Since u is continuous on B_r (and so is v on D_r), by Sobolev embedding theorem there exists some constant $C > 0$ such that

$$\|v_j - v\|_{C(D_r)} \le \|u_j - u\|_{C(B_r)} \le C \|u_j - u\|_{W^{1,p}(B_r)} \to 0$$

as $j \to 0$. In particular,

$$v_j \to v \quad \text{on} \quad D_r \tag{3.4}$$

in the sense of distributions.

On the other hand, applying Lemma 3.1 to $|\nabla u - \nabla u_j|^p$, the function

$$g_j(\zeta) := \int_{|w|<r} |w|^{2n-2} |\nabla u(w\zeta) - \nabla u_j(w\zeta)|^p dv_w, \quad \zeta \in S^{2n-1}$$

satisfies

$$\begin{aligned}\int_{|\zeta|=1} |g_j(\zeta)| dS_\zeta &= \int_{|\zeta|=1} \int_{|w|<r} |w|^{2n-2} |\nabla u(w\zeta) - \nabla u_j(w\zeta)|^p dv_w dS_\zeta \\ &= 2\pi \int_{|z|<r} |\nabla u(z) - \nabla u_j(z)|^p dv_z \to 0\end{aligned}$$

as $j \to \infty$. Hence, by passing to a subsequence if necessary (see, for instance, [3, Theorem 4.9]), one has for a.e. $\zeta \in S^{2n-1}$,

$$g_j(\zeta) \to 0$$

as $j \to 0$. Making use of Hölder's inequality

$$\begin{aligned}\int_{|w|<r} |\zeta \cdot \nabla u_j(w\zeta) - \zeta \cdot \nabla u(w\zeta)| dv_w &\le \int_{|w|<r} |w|^{-\frac{2n-2}{p}} \cdot |w|^{\frac{2n-2}{p}} |\nabla u_j(w\zeta) - \nabla u(w\zeta)| dv_w \\ &\le \left(\int_{|w|<r} |w|^{-\frac{2n-2}{p-1}}\right)^{\frac{p-1}{p}} \cdot (g_j(\zeta))^{\frac{1}{p}} \to 0\end{aligned}$$

as $j \to \infty$. Here we used the fact that $p > 2n$, so $\frac{2n-2}{p-1} < 2$ particularly. This implies that

$$\zeta \cdot \nabla u_j(w\zeta) \to \zeta \cdot \nabla u(w\zeta) \quad \text{on} \quad D_r \tag{3.5}$$

in the sense of distributions. (3.2) is thus proved in view of (3.3), (3.4) and (3.5). □

The following lemma allows us to extend the flatness of $W^{1,p}_{loc}, p > 2n$ solutions to their restrictions along the radial directions.

Lemma 3.4. *Suppose $u \in W^{1,p}(B_r)$ for some $p > 2n(= \dim_{\mathbb{R}} B_r)$. Then u vanishes to infinite order in the L^1 sense at 0 if and only if for every $m \ge 1$,*

$$|u(z)| = O(|z|^m) \quad \text{for all} \quad |z| << 1. \tag{3.6}$$

In particular, if u vanishes to infinite order in the L^1 sense at $0 \in B_r$, then for a.e. $\zeta \in S^{2n-1}$, $v(w) := u(w\zeta)$ as a function of $w \in D_r$ vanishes to infinite order in the L^2 sense at $0 \in D_r$.

Proof. The backward direction is obvious by definition. In fact, one can also easily check that, for any given $q \ge 1$, as long as $u \in L^q$ near 0 and satisfies (3.6), u vanishes to infinite order in the L^q sense at 0.

Assume u vanishes to infinite order in the L^1 sense at 0. Since $p > 2n$, by Sobolev embedding theorem u is continuous on B_r with

$$\sup_{|z|<r} |u| + \int_{|z|<r} |\nabla u|^p \le C_0. \tag{3.7}$$

for some constant $C_0 > 0$. Fix some q with $2n < q < p$. For every $m \ge 1$, by the continuity of u and the L^1 flatness of u at 0, one has $u(0) = 0$ and

$$\int_{|z|<t} |u|^{\frac{qp}{p-q}} \le C_0^{\frac{qp}{p-q}-1} \int_{|z|<t} |u| \le O(t^{\frac{2pqm}{p-q}}), \quad \text{for all} \quad t << 1. \tag{3.8}$$

Letting $v := u^2$, then $v \in W^{1,p}(B_r)$ with $\nabla v = 2u\nabla u$ on B_r. Moreover, making use of Hölder's inequality, (3.7) and (3.8),

$$\begin{aligned}\int_{|z|<t} |\nabla v|^q =& 2^q \int_{|z|<t} |u|^q |\nabla u|^q \le 2^q \left(\int_{|z|<t} |u|^{\frac{qp}{p-q}}\right)^{\frac{p-q}{p}} \left(\int_{|z|<t} |\nabla u|^p\right)^{\frac{q}{p}} \\ \le& 2^p C_0^{\frac{q}{p}} \left(\int_{|z|<t} |u|^{\frac{qp}{p-q}}\right)^{\frac{p-q}{p}} \le O(t^{2qm}), \quad \text{for all} \quad t << 1.\end{aligned} \tag{3.9}$$

On the other hand, since $v(0)=0$ and $q>2n$, there exists a constant $C_1>0$ such that

$$\sup_{|z|\le \frac{t}{2}} |v| \le C_1 t^{1-\frac{2n}{q}} \left(\int_{B_t} |\nabla v|^q\right)^{\frac{1}{q}}, \quad \text{for all} \quad t<r.$$

See, for instance, [5, pp. 283]. Together with (3.9), we get

$$|u(z)|^2 = |v(z)| \le C_1 \left(\int_{B_{2|z|}} |\nabla v|^q\right)^{\frac{1}{q}} \le O(|z|^{2m}), \quad \text{for all} \quad |z| << 1.$$

This proves the forward direction.

Finally, if u vanishes to infinite order in the L^1 sense at 0, then u satisfies (3.6) by the equivalence of the two types of flatness. Hence for a.e. $\zeta\in S^{2n-1}$, (3.6) holds true for v as well. In particular, since $v\in L^2$ near 0 due to Lemma 3.3, v vanishes to infinite order in the L^2 sense at 0. □

We would like to note that, although both Lemma 3.3 and Lemma 3.4 are stated for scalar functions for simplicity of notations, they can be seamlessly extended to the case of vector-valued functions.

4 Proof of the main theorems

In this section we shall prove Theorems 1.1-1.4. Let us start by stating a local ellipticity lemma of $\bar\partial$, which will be repeatedly used in the boot-strap argument.

Lemma 4.1. *[8, Lemma 3.1] Let Ω be a domain in $\mathbb{C}^n$ and $1<p<\infty$. Let $V\in L^p_{loc}(\Omega)$ be a $\bar\partial$-closed $(0,1)$ form on Ω. Then every solution to $\bar\partial f = V$ on Ω in the sense of distributions belongs to $W^{1,p}_{loc}(\Omega)$.*

Proof of Theorem 1.1: Without loss of generality, let $z_0=0$, and r be small such that $B_r\subset\Omega$. Since $u\in L^{\frac{2n}{2n-1}}_{loc}(B_r)$ by Sobolev embedding theorem, as an application of Hölder's inequality, $Vu\in L^{\frac{2n}{(2n-1)+\frac{2n}{p}}}_{loc}(B_r)$. It follows from Lemma 4.1 and the inequality (1.1) that $u\in W^{1,\frac{2n}{(2n-1)+\frac{2n}{p}}}_{loc}(B_r)\subset L^{\frac{2n}{(2n-1)+\frac{2n}{p}-1}}_{loc}(B_r)$. Since $p>2n$, a boot-strap argument as above can eventually give $u\in W^{1,p}_{loc}(B_r)$.

For each fixed $\zeta\in S^{2n-1}$, let $v(w):=u(w\zeta)$ and $\tilde V(w):=V(w\zeta), w\in D_r$. Since $V\in L^p_{loc}(B_r), p>2n$, by Corollary 3.2 we have $\tilde V\in L^2_{loc}(D_r)$. On the other hand, it follows from Lemma 3.4 and the L^1 flatness of u at 0 that v vanishes to infinite order in the L^2 sense at 0. Moreover, $v\in W^{1,2}_{loc}(D_r)$ by Lemma 3.3, and as a consequence of (3.2),

$$|\bar\partial v(w)| = |\zeta\cdot\bar\partial u(w\zeta)| \le V(w\zeta)|u(w\zeta)| = \tilde V(w)|v(w)|, \quad w\in D_r.$$

Hence we can make use of Theorem 2.1 part 2) to v and obtain $v=0$ on D_r for a.e. $\zeta\in S^{2n-1}$. Thus $u=0$ on B_r. Apply the weak unique continuation property in Theorem 2.1 part 1) to further get $u\equiv 0$ on Ω. □

Before proving Theorems 1.2-1.3 for L^{2n}_{loc} potentials, we point out that the slicing method in the proof of Theorem 1.1 fails to work in general if V is merely in L^{2n}_{loc}. More precisely, there exists a L^{2n}_{loc} potential V whose complex radial restriction is not in L^2_{loc}.

Example 3. *Assume $n \geq 2$. For each $\epsilon \in \left(\frac{1}{2}, \frac{2n-1}{2n}\right)$, consider*

$$u(z) = e^{-(-\ln|z|)^{\epsilon}}, \quad z \in B_{\frac{1}{2}} \subset \mathbb{C}^n,$$

and

$$V(z) = \frac{\epsilon(-\ln|z|)^{\epsilon-1}}{2|z|}, \quad z \in B_{\frac{1}{2}}.$$

Then $u \in W^{1,2n}(B_{\frac{1}{2}})$, $V \in L^{2n}(B_{\frac{1}{2}})$ and

$$|\bar{\partial} u| \leq V|u| \quad on \quad B_{\frac{1}{2}}.$$

(One can verify that u only vanishes to a finite order at 0 in the L^q sense for every $q \geq 1$.) On the other hand, for each $\zeta \in S^{2n-1}$, the complex radial restriction of V is

$$\tilde{V}(w) := V(w\zeta) = \frac{\epsilon(-\ln|w|)^{\epsilon-1}}{2|w|}, \quad w \in D_{\frac{1}{2}}.$$

It is easy to see that $\tilde{V} \notin L^2_{loc}(D_{\frac{1}{2}})$ since $\epsilon > \frac{1}{2}$.

Proof of Theorems 1.2 and 1.3: Let $z_0 = 0$ and $r > 0$ be small such that $B_r \subset \Omega$. For each fixed $\zeta \in S^{2n-1}$, let $\tilde{V}(w) := |w|^{\frac{n-1}{n}} V(w\zeta)$ and $v(w) := u(w\zeta), w \in D_r$. It follows from Lemma 3.1 that $\tilde{V} \in L^{2n}_{loc}(D_r)$ for a.e. $\zeta \in S^{2n-1}$. On the other hand, since $u \in W^{1,p}_{loc}(B_r)$ for some $p > 2n$, v vanishes to infinite order at 0 in the L^2 sense by Lemma 3.4. Moreover, as a consequence of Lemma 3.3, $v \in W^{1,2}_{loc}(D_r)$ and satisfies

$$|\bar{\partial} v(w)| \leq |w|^{-\frac{n-1}{n}} \tilde{V}(w)|v(w)|, \quad w \in D_r.$$

For a.e. $\zeta \in S^{2n-1}$, employ Theorem 2.4 if $N = 1$; employ Theorem 2.5 if $n = 2$. In both cases, we get $v = 0$ on D_r and thus $u = 0$ on B_r. Apply the weak unique continuation property to get $u \equiv 0$. □

Remark 4.2. *In view of Theorems 1.1-1.3, the following two questions still remain open. With an approach similar as in the proof to Theorems 1.2-1.3, the resolution of Question 1 is reduced to that of Question 2.*

1. *Let Ω be a domain in $\mathbb{C}^n, n \geq 3$ and $N \geq 2$. Suppose $u : \Omega \to \mathbb{C}^N$ with $u \in W^{1,p}_{loc}(\Omega)$ for some $p > 2n$ and satisfies $|\bar{\partial} u| \leq V|u|$ a.e. on Ω for some $V \in L^{2n}_{loc}(\Omega)$. If u vanishes to infinite order in the L^1 sense at some $z_0 \in \Omega$, does u vanish identically?*

2. *Let Ω be a domain in $\mathbb{C}$ containing 0, and $n, N \in \mathbb{Z}^+$ with $n \geq 3, N \geq 2$. Suppose $u : \Omega \to \mathbb{C}^N$ with $u \in W^{1,2}_{loc}(\Omega)$ and satisfies $|\bar{\partial} u| \leq |z|^{-\frac{n-1}{n}} V|u|$ a.e. on Ω for some $V \in L^{2n}_{loc}(\Omega)$. If u vanishes to infinite order in the L^2 sense at $0 \in \Omega$, does u vanish identically?*

Next we prove Theorem 1.4 for $W^{1,1}_{loc}$ solutions to $|\bar{\partial} u| \leq \frac{C}{|z|}|u|$ a.e. on Ω, where C is a positive constant. It is not hard to see that $u \in W^{1,q}$ everywhere away from 0 for all $q < \infty$ using a boot-strap argument since $\frac{1}{|z|} \in L^\infty$ off 0. However, the argument is not directly effective near 0 because $\frac{1}{|z|} \notin L^{2n}$ there. The following two lemmas show that $u \in W^{1,q}$ for all $q < \infty$ near 0 under the L^1 flatness assumption at 0.

Lemma 4.3. *Let $u \in L^1$ near $0 \in \mathbb{C}^n$, and vanishes to infinite order in the L^1 sense at 0. Then for each $M > 0$, $\frac{u}{|z|^M} \in L^1$ near 0, and vanishes to infinite order in the L^1 sense at 0.*

Proof. For each $m \geq 1$ and $\epsilon > 0$, by the L^1 flatness of u at 0,

$$\int_{|z|<r} |u| dv_z \leq \epsilon r^{m+M} \quad \text{for all} \quad r << 1.$$

Then

$$\begin{aligned}\int_{|z|<r} \frac{|u|}{|z|^M} dv_z &= \sum_{j=1}^{\infty} \int_{\frac{r}{2^j}<|z|<\frac{r}{2^{j-1}}} \frac{|u|}{|z|^M} dv_z \leq \sum_{j=1}^{\infty} \frac{2^{Mj}}{r^M} \int_{\frac{r}{2^j}<|z|<\frac{r}{2^{j-1}}} |u| dv_z \leq \sum_{j=1}^{\infty} \frac{2^{Mj}}{r^M} \int_{|z|<\frac{r}{2^{j-1}}} |u| dv_z \\ &\leq \epsilon \sum_{j=1}^{\infty} \frac{2^{Mj}}{r^M} \frac{r^{m+M}}{2^{(m+M)(j-1)}} = \epsilon 2^M r^m \sum_{j=1}^{\infty} 2^{-m(j-1)} \leq \epsilon 2^{M+1} r^m \quad \text{for all} \quad r << 1.\end{aligned}$$

In particular, $\frac{u}{|z|^M} \in L^1$ near 0, and vanishes to infinite order in the L^1 sense at 0. □

Lemma 4.4. *Let Ω be a domain in $\mathbb{C}^n$ and $0 \in \Omega$. Let $u : \Omega \to \mathbb{C}^N$ with $u \in W^{1,1}_{loc}(\Omega)$, and satisfy $|\bar{\partial} u| \leq \frac{C}{|z|}|u|$ a.e. on Ω for some constant $C > 0$. Assume u vanishes to infinite order in the L^1 sense at $0 \in \Omega$. Then $u \in W^{1,q}_{loc}(\Omega)$ for every $q < \infty$.*

Proof. We first claim if $u \in L^p$ near 0 for some $p > 1$, then $\frac{u}{|z|} \in L^q$ for every $1 < q < p$. Indeed, for each $q < p$, letting $\epsilon = \frac{p-q}{p-1}$, then $0 < \epsilon < 1$ and $\frac{q-\epsilon}{1-\epsilon} = p$. By Hölder's inequality and Lemma 4.3,

$$\int_{|z|<r} \left|\frac{u}{|z|}\right|^q = \int_{|z|<r} \frac{|u|^\epsilon}{|z|^q} \cdot |u|^{q-\epsilon} \leq \left(\int_{|z|<r} \frac{|u|}{|z|^{\frac{q}{\epsilon}}}\right)^\epsilon \left(\int_{|z|<r} |u|^p\right)^{1-\epsilon} < \infty, \quad r << 1.$$

The claim is proved.

We are now ready to employ the boot-strap argument as in the proof to Theorem 1.1. Since $u \in W^{1,1}$ near 0, it follows from the Sobolev embedding theorem that $u \in L^{\frac{2n}{2n-1}}$ near 0. Consequently, the above proved claim gives $\frac{u}{|z|} \in L^q$ near 0 for any $q < \frac{2n}{2n-1}$. Lemma 4.1 further allows us to obtain $u \in W^{1,q} \subset L^{q'}$ near 0 for any $q' < \frac{2n}{2n-2}$. Repeating the process eventually leads to $u \in W^{1,q}$ near 0 for every $q < \infty$. The fact that $u \in W^{1,q}$ near every other point than 0 is proved in a similar but simpler manner (without using the claim) since $\frac{1}{|z|} \in L^\infty$ near those points. □

Proof of Theorem 1.4: The $N = 1$ case is an immediate consequence of Theorem 2.2 and Lemma 4.4. So we assume $N \geq 2$. Let r be small such that $B_r \subset \Omega$. For each fixed $\zeta \in S^{2n-1}$, let $v(w) := u(w\zeta), w \in D_r$. Making use of Lemma 4.4, Lemma 3.3 and Lemma 3.4, we have $v \in W^{1,2}_{loc}(D_r)$, vanishes to infinite order in the L^2 sense at 0 and satisfies

$$|\bar{\partial} v(w)| = |\zeta \cdot \bar{\partial} u(w\zeta)| \leq \frac{C}{|w|}|u(w\zeta)| = \frac{C}{|w|}|v(w)|, \quad w \in D_r.$$

Thus, for a.e. $\zeta \in S^{2n-1}$ we can apply Theorem 2.3 to get $v = 0$ on D_r. Hence $u = 0$ on B_r. The weak unique continuation property further applies to give $u \equiv 0$. □

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pan@pfw.edu,

Department of Mathematical Sciences, Purdue University Fort Wayne, Fort Wayne, IN 46805-1499, USA.

zhangyu@pfw.edu,

Department of Mathematical Sciences, Purdue University Fort Wayne, Fort Wayne, IN 46805-1499, USA.